ASSIGNMENT 2 — OPERATIONAL SEMANTICS

COMP 3010 — ORGANIZATION OF PROGRAMMING LANGUAGES

1. Big-Step Operational Semantics

For exercises 1 to 3, consider this definition of conditional arithmetic in terms of a big-step operational semantics.

Syntax of arithmetic (A) and boolean (B) expressions, and natural number (n) and boolean (b) values:

$$n ::= 0 \mid 1 \mid 2 \mid 3 \mid \dots$$

$$A ::= \underline{n} \mid \texttt{plus}(A_1, A_2) \mid \texttt{minus}(A_1, A_2) \mid \texttt{times}(A_1, A_2) \mid \texttt{div}(A_1, A_2) \mid \texttt{if}(B, A_1, A_2)$$

$$b ::= \mathit{true} \mid \mathit{false}$$

$$B ::= \underline{b} \mid \texttt{and}(B_1, B_2) \mid \texttt{or}(B_1, B_2) \mid \texttt{zero?}(A)$$

Big-step operational semantics of arithmetic expressions ($A \Downarrow n$):

$$\frac{}{\underline{n} \Downarrow n}$$

$$\frac{A_1 \Downarrow n_1 \quad A_2 \Downarrow n_2 \quad n_1 + n_2 = n}{\texttt{plus}(A_1, A_2) \Downarrow n} \qquad \frac{A_1 \Downarrow n_1 \quad A_2 \Downarrow n_2 \quad n_1 - n_2 = n \quad n_1 \geq n_2}{\texttt{minus}(A_1, A_2) \Downarrow n}$$

$$\frac{A_1 \Downarrow n_1 \quad A_2 \Downarrow n_2 \quad n_1 \times n_2 = n}{\texttt{times}(A_1, A_2) \Downarrow n} \qquad \frac{A_1 \Downarrow n_1 \quad A_2 \Downarrow n_2 \quad n_1 \div n_2 = n \quad n_2 \neq 0}{\texttt{div}(A_1, A_2) \Downarrow n}$$

$$\frac{B \Downarrow \texttt{true} \quad A_1 \Downarrow n_1}{\texttt{if}(B, A_1, A_2) \Downarrow n_1} \qquad \frac{B \Downarrow \texttt{false} \quad A_2 \Downarrow n_2}{\texttt{if}(B, A_1, A_2) \Downarrow n_2}$$

Big-step operational semantics of boolean expressions ($B \Downarrow b$):

$$\frac{}{\underline{\mathit{true}} \Downarrow \mathit{true}} \qquad \frac{}{\underline{\mathit{false}} \Downarrow \mathit{false}}$$

$$\frac{B_1 \Downarrow \mathit{true} \quad B_2 \Downarrow b}{\texttt{and}(B_1, B_2) \Downarrow b} \qquad \frac{B_1 \Downarrow \mathit{false}}{\texttt{and}(B_1, B_2) \Downarrow \mathit{false}}$$

$$\frac{B_1 \Downarrow \mathit{false} \quad B_2 \Downarrow b}{\texttt{or}(B_1, B_2) \Downarrow b} \qquad \frac{B_1 \Downarrow \mathit{true}}{\texttt{or}(B_1, B_2) \Downarrow \mathit{true}}$$

$$\frac{A \Downarrow 0}{\texttt{zero?}(A) \Downarrow \mathit{true}} \qquad \frac{A \Downarrow n \quad n \neq 0}{\texttt{zero?}(A) \Downarrow \mathit{false}}$$

For the natural number division $n_1 \div n_2$ returns only the whole number dividend and drops the remainder, so that $7 \div 2$ is 3 for example.

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Exercise 1 (Multiple Choice)**.** Which of the following evaluations of

$$\texttt{times}\left(\texttt{if}\left(\texttt{or}\left(\texttt{zero?}(\underline{0}), \texttt{zero?}(\texttt{div}(\underline{0},\underline{1}))\right), \underline{4}, \underline{2}\right), \texttt{minus}(\underline{5},\underline{3})\right)$$

can be derived by the operational semantics?

(a) $\texttt{times}\left(\texttt{if}\left(\texttt{or}\left(\texttt{zero?}(\underline{0}), \texttt{zero?}(\texttt{div}(\underline{0},\underline{1}))\right), \underline{4}, \underline{2}\right), \texttt{minus}(\underline{5},\underline{3})\right) \Downarrow 0$

(b) $\texttt{times}\left(\texttt{if}\left(\texttt{or}\left(\texttt{zero?}(\underline{0}), \texttt{zero?}(\texttt{div}(\underline{0},\underline{1}))\right), \underline{4}, \underline{2}\right), \texttt{minus}(\underline{5},\underline{3})\right) \Downarrow 2$

(c) $\texttt{times}\left(\texttt{if}\left(\texttt{or}\left(\texttt{zero?}(\underline{0}), \texttt{zero?}(\texttt{div}(\underline{0},\underline{1}))\right), \underline{4}, \underline{2}\right), \texttt{minus}(\underline{5},\underline{3})\right) \Downarrow 4$

(d) $\texttt{times}\left(\texttt{if}\left(\texttt{or}\left(\texttt{zero?}(\underline{0}), \texttt{zero?}(\texttt{div}(\underline{0},\underline{1}))\right), \underline{4}, \underline{2}\right), \texttt{minus}(\underline{5},\underline{3})\right) \Downarrow 8$

(e) $\texttt{times}\left(\texttt{if}\left(\texttt{or}\left(\texttt{zero?}(\underline{0}), \texttt{zero?}(\texttt{div}(\underline{0},\underline{1}))\right), \underline{4}, \underline{2}\right), \texttt{minus}(\underline{5},\underline{3})\right) \Downarrow \mathit{true}$

(f) $\texttt{times}\left(\texttt{if}\left(\texttt{or}\left(\texttt{zero?}(\underline{0}), \texttt{zero?}(\texttt{div}(\underline{0},\underline{1}))\right), \underline{4}, \underline{2}\right), \texttt{minus}(\underline{5},\underline{3})\right) \Downarrow \mathit{false}$

Exercise 2 (This or That)**.** An arithmetic expression A *returns* if there is some number n such that $A \Downarrow n$, and *diverges* if there is no such n. Similarly, a boolean expression B *returns* if there is some boolean value b such that $B \Downarrow b$, and *diverges* otherwise.

For each of the following arithmetic and boolean expressions, say if that expression returns or diverges and reason your way through using a derivation tree for each.

(a) $\texttt{div}\left(\texttt{plus}(\underline{3},\underline{1}),\ \texttt{minus}(\underline{5},\underline{5})\right)$

(b) $\texttt{if}\left(\texttt{and}\left(\underline{\mathit{false}}, \texttt{zero?}(\texttt{div}(\underline{1},\underline{0}))\right),\ \texttt{div}(\underline{3},\underline{0}),\ \underline{7}\right)$

(c) $\texttt{and}\left(\texttt{zero?}(\texttt{minus}(\underline{2},\underline{3})),\ \underline{\mathit{true}}\right)$

(d) $\texttt{or}\left(\underline{\mathit{true}},\ \texttt{zero?}(\texttt{div}(\underline{0},\underline{0}))\right)$

(e) $\texttt{if}\left(\texttt{zero?}(\underline{0}),\ \texttt{div}(\underline{10},\underline{2}),\ \texttt{plus}(\underline{1},\texttt{div}(\underline{0},\underline{0}))\right)$

Exercise 3 (Show Your Work)**.** Determine the number the following expression evaluates to by drawing a derivation tree of the big-step operational semantics:

$$\texttt{if}\left(\texttt{and}\left(\texttt{zero?}(\texttt{minus}(\texttt{plus}(\underline{2},\underline{2}),\underline{4})),\ \underline{\mathit{true}}\right), \texttt{div}(\texttt{times}(\underline{6},\underline{3}),\underline{3}), \texttt{div}(\underline{5},\texttt{minus}(\underline{2},\underline{2}))\right)$$

2. Small-Step Operational Semantics

For exercises 4 to 6, consider this definition of a small-step operational semantics for the same conditional arithmetic language used previously in section 1.

Small-step reduction rules:

$$\begin{array}{rlrl}
\texttt{plus}(\underline{n_1},\underline{n_2}) \mapsto \underline{n} & (n = n_1 + n_2) & \texttt{minus}(\underline{n_1},\underline{n_2}) \mapsto \underline{n} & (n = n_1 - n_2,\ n_1 \geq n_2) \\
\texttt{times}(\underline{n_1},\underline{n_2}) \mapsto \underline{n} & (n = n_1 \times n_2) & \texttt{div}(\underline{n_1},\underline{n_2}) \mapsto \underline{n} & (n = n_1 \div n_2,\ n_2 \neq 0) \\
\texttt{if}(\underline{\mathit{true}}, A_1, A_2) \mapsto A_1 & & \texttt{if}(\underline{\mathit{false}}, A_1, A_2) \mapsto A_2 & \\
\texttt{and}(\underline{\mathit{true}}, B) \mapsto B & & \texttt{and}(\underline{\mathit{false}}, B) \mapsto \underline{\mathit{false}} & \\
\texttt{or}(\underline{\mathit{false}}, B) \mapsto B & & \texttt{or}(\underline{\mathit{true}}, B) \mapsto \underline{\mathit{true}} & \\
\texttt{zero?}(\underline{0}) \mapsto \underline{\mathit{true}} & & \texttt{zero?}(\underline{n}) \mapsto \underline{\mathit{false}} & (n \neq 0)
\end{array}$$

Evaluation contexts (E):

$$\begin{aligned} E ::= {} & \square \mid \mathtt{plus}(E, A) \mid \mathtt{plus}(\underline{n}, E) \mid \mathtt{minus}(E, A) \mid \mathtt{minus}(\underline{n}, E) \\ & \mid \mathtt{times}(E, A) \mid \mathtt{times}(\underline{n}, E) \mid \mathtt{div}(E, A) \mid \mathtt{div}(\underline{n}, E) \mid \mathtt{if}(E, A_1, A_2) \\ & \mid \mathtt{and}(E, B) \mid \mathtt{or}(E, B) \mid \mathtt{zero?}(E) \end{aligned}$$

Reducing sub-expressions is *only* allowed within evaluation contexts:

$$\frac{A \mapsto A}{E[A] \mapsto E[A']} \qquad\qquad \frac{A \mapsto B'}{E[B] \mapsto E[B']}$$

Exercise 4 (Multiple Choice)**.** What do you get from plugging the expression $\mathtt{minus}(\mathtt{plus}(\underline{2}, \underline{1}), \underline{3})$ into the evaluation context $\mathtt{if}(\mathtt{zero?}(\square), \mathtt{times}(\underline{3}, \underline{5}), \mathtt{plus}(\underline{1}, \underline{1}))$?

(a) $\mathtt{if}(\mathtt{zero?}(\mathtt{minus}(\mathtt{plus}(\underline{2}, \underline{1}), \underline{3})), \mathtt{times}(\underline{3}, \underline{5}), \mathtt{plus}(\underline{1}, \underline{1}))$
(b) $\mathtt{if}(\mathtt{zero?}(\underline{0}), \mathtt{times}(\underline{3}, \underline{5}), \mathtt{plus}(\underline{1}, \underline{1}))$
(c) $\mathtt{if}(\mathtt{minus}(\mathtt{plus}(\underline{2}, \underline{1}), \underline{3}), \mathtt{times}(\underline{3}, \underline{5}), \mathtt{plus}(\underline{1}, \underline{1}))$
(d) $\mathtt{if}(\mathtt{zero?}(\mathtt{plus}(\underline{2}, \underline{1})), \mathtt{times}(\underline{3}, \underline{5}), \mathtt{minus}(\underline{3}, \underline{3}))$

Exercise 5 (This or That)**.** For each of the following pairs of a context and a sub-expression, identify which ones are valid or invalid decompositions of the expression

$$\mathtt{if}(\mathtt{zero?}(\mathtt{times}(\mathtt{plus}(\underline{3}, \underline{4}), \mathtt{minus}(\underline{1}, \underline{1}))), \mathtt{minus}(\underline{5}, \underline{3}), \mathtt{div}(\underline{6}, \underline{2}))$$

according to the grammar of E in small-step operational semantics. (There may be multiple valid decompositions.)

(a) $\mathtt{if}(\square, \mathtt{minus}(\underline{5}, \underline{3}), \mathtt{div}(\underline{6}, \underline{2}))$ **and** $\mathtt{zero?}(\mathtt{times}(\mathtt{plus}(\underline{3}, \underline{4}), \mathtt{minus}(\underline{1}, \underline{1})))$
(b) $\mathtt{if}(\mathtt{zero?}(\mathtt{times}(\mathtt{plus}(\underline{3}, \underline{4}), \mathtt{minus}(\underline{1}, \underline{1}))), \square, \mathtt{div}(\underline{6}, \underline{2}))$ **and** $\mathtt{minus}(\underline{5}, \underline{3})$
(c) $\mathtt{if}(\mathtt{zero?}(\mathtt{times}(\mathtt{plus}(\underline{3}, \underline{4}), \mathtt{minus}(\underline{1}, \underline{1}))), \mathtt{minus}(\underline{5}, \underline{3}), \square)$ **and** $\mathtt{div}(\underline{6}, \underline{2})$
(d) $\mathtt{if}(\mathtt{zero?}(\mathtt{times}(\square, \mathtt{minus}(\underline{1}, \underline{1}))), \mathtt{minus}(\underline{5}, \underline{3}), \mathtt{div}(\underline{6}, \underline{2}))$ **and** $\mathtt{plus}(\underline{3}, \underline{4})$
(e) $\mathtt{if}(\mathtt{zero?}(\mathtt{times}(\mathtt{plus}(\underline{3}, \underline{4}), \square)), \mathtt{minus}(\underline{5}, \underline{3}), \mathtt{div}(\underline{6}, \underline{2}))$ **and** $\mathtt{minus}(\underline{1}, \underline{1})$

Exercise 6 (Show Your Work)**.** Write down the sequence of reduction steps using the rules of the small-step operational semantics for simplifying to its final result.

$$\mathtt{if}\left(\mathtt{zero?}\left(\mathtt{times}(\mathtt{plus}(\underline{1}, \underline{2}), \mathtt{minus}(\underline{3}, \underline{3}))\right), \mathtt{div}(\underline{7}, \underline{3}), \mathtt{minus}(\underline{8}, \underline{2})\right)$$